

# CLASSIFYING LOCALLY COMPACT SEMITOPOLOGICAL POLYCYCLIC MONOIDS

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**ABSTRACT.** We present a complete classification of Hausdorff locally compact polycyclic monoids up to a topological isomorphism. A *polycyclic monoid* is an inverse monoid with zero, generated by a subset  $\Lambda$  such that  $xx^{-1} = 1$  for any  $x \in \Lambda$  and  $xy^{-1} = 0$  for any distinct  $x, y \in \Lambda$ . We prove that any non-discrete Hausdorff locally compact topology with continuous shifts on a polycyclic monoid  $M$  coincides with the topology of one-point compactification of the discrete space  $M \setminus \{0\}$ .

## INTRODUCTION

In this paper we present a complete classification of locally compact semitopological polycyclic monoids up to a topological isomorphism.

We shall follow the terminology of [8, 10, 19, 22]. First we recall some information on inverse semigroups and monoids. We identify cardinals with the sets of ordinals of smaller cardinality.

A *semigroup* is a set  $S$  endowed with an associative binary operation  $\cdot : S \times S \rightarrow S$ ,  $\cdot : (x, y) \mapsto xy$ . An element  $e \in S$  is called the *unit* (resp. *zero*) of  $S$  if  $xe = x = ex$  (resp.  $xe = e = ex$ ) for all  $x \in S$ . A semigroup can contain at most one unit (which will be denoted by 1) and at most one zero (denoted by 0). A *monoid* is a semigroup with a unit.

A semigroup  $S$  is called *inverse* if for every element  $a \in S$  there exists a unique element  $a^{-1}$  (called the *inverse* of  $a$ ) such that  $aa^{-1}a = a$  and  $a^{-1}aa^{-1} = a^{-1}$ . An *inverse monoid* is an inverse semigroup with unit. We say that an inverse monoid  $S$  is *generated* by a subset  $\Lambda \subset S$  if  $S$  coincides with the smallest subsemigroup of  $S$  containing the set  $\Lambda \cup \Lambda^{-1}$ .

A *polycyclic monoid* is an inverse monoid  $S$  with zero  $0 \neq 1$ , which is generated by a subset  $\Lambda \subset S$  such that  $xx^{-1} = 1$  for all  $x \in \Lambda$  and  $xy^{-1} = 0$  for any distinct  $x, y \in \Lambda$ . If the generating set  $\Lambda$  has cardinality  $\lambda$ , then  $S$  is called a  $\lambda$ -*polycyclic monoid*. We claim that  $|\Lambda| \geq 2$ . In the opposite case,  $\Lambda = \{x\}$  is a singleton and  $0 \in S = \{x^{-n}x^m : n, m \in \omega\}$ , which implies that  $0 = x^{-n}x^m$  for some non-negative numbers  $n, m$ . Then  $0 = x^{n+1} \cdot 0 \cdot x^{-m} = x^{n+1}(x^{-n}x^m)x^{-m} = x$  and hence  $1 = xx^{-1} = 0x^{-1} = 0$ , but this contradicts the definition of a polycyclic monoid.

A canonical example of a  $\lambda$ -polycyclic monoid can be constructed as follows. Let  $\mathcal{M}_{\lambda^{\pm}}$  be the monoid of all words in the alphabet  $\{x, x^{-1} : x \in \lambda\}$ , endowed with the semigroup operation of concatenation of words. The empty word is the unit 1 of the monoid  $\mathcal{M}_{\lambda^{\pm}}$ . Let  $\mathcal{M}_{\lambda^{\pm}}^0 := \mathcal{M}_{\lambda^{\pm}} \cup \{0\}$  be the monoid  $\mathcal{M}_{\lambda^{\pm}}$  with the attached external zero, i.e., an element  $0 \notin \mathcal{M}_{\lambda^{\pm}}$  such that  $0 \cdot x = 0 = x \cdot 0$  for all  $x \in \mathcal{M}_{\lambda^{\pm}}^0$ . On the monoid  $\mathcal{M}_{\lambda^{\pm}}^0$  consider the smallest congruence  $\sim$  containing the pairs  $(xx^{-1}, 1)$  and  $(xy^{-1}, 0)$  for all distinct elements  $x, y \in \lambda$ . Then the quotient semigroup  $\mathcal{M}_{\lambda^{\pm}}^0 / \sim$  is the required canonical example of a  $\lambda$ -polycyclic monoid, which will be denoted by  $\mathcal{P}_{\lambda}$  and called *the  $\lambda$ -polycyclic monoid*.

Algebraic properties of the  $\lambda$ -polycyclic monoid were deeply investigated in [5]. According to [5, Theorem 2.5], the semigroup  $\mathcal{P}_{\lambda}$  is congruence-free, which implies that each  $\lambda$ -polycyclic monoid is algebraically isomorphic to  $\mathcal{P}_{\lambda}$ .

The aim of this paper is to describe Hausdorff locally compact topologies on  $\mathcal{P}_{\lambda}$ , compatible with the algebraic structure of the semigroup  $\mathcal{P}_{\lambda}$ . A suitable compatibility condition is given by the notion of a semitopological semigroup.

*Date:* November 23, 2016.

*2010 Mathematics Subject Classification.* Primary 20M18, 22A15. Secondary 55N07.

*Key words and phrases.* locally compact semitopological semigroup,  $\alpha$ -polycyclic monoid, bicyclic semigroup.

A *semitopological semigroup* is a semigroup  $S$  endowed with a Hausdorff topology making the binary operation  $S \times S \rightarrow S$ ,  $(x, y) \mapsto xy$ , separately continuous. If this operation is jointly continuous, then  $S$  is called a *topological semigroup*.

For a cardinal  $\lambda \geq 2$  by  $\mathcal{P}_\lambda^d$  we shall denote the  $\lambda$ -polycyclic monoid  $\mathcal{P}_\lambda$  endowed with the discrete topology and by  $\mathcal{P}_\lambda^c$  the monoid  $\mathcal{P}_\lambda$  endowed with the compact topology  $\tau = \{U \subset \mathcal{P}_\lambda : 0 \in U \Rightarrow (\mathcal{P}_\lambda \setminus U \text{ is finite})\}$  of one-point compactification of the discrete space  $\mathcal{P}_\lambda \setminus \{0\}$ . It is clear that  $\mathcal{P}_\lambda^d$  is a topological monoid. On the other hand,  $\mathcal{P}_\lambda^c$  is a compact semitopological monoid, which is not a topological semigroup.

By [5], each locally compact topological  $\lambda$ -polycyclic monoid is discrete and hence is topologically isomorphic to  $\mathcal{P}_\lambda^d$ . In the semitopological case we have the following dichotomy, which is the main result of this paper.

**Main Theorem.** Any locally compact semitopological polycyclic monoid  $S$  is either discrete or compact. More precisely,  $S$  is topologically isomorphic either to  $\mathcal{P}_\lambda^d$  or to  $\mathcal{P}_\lambda^c$  for a unique cardinal  $\lambda \geq 2$ .

Since the compact semitopological  $\lambda$ -polycyclic monoid  $\mathcal{P}_\lambda^c$  fails to be a topological semigroup, Main Theorem implies the mentioned result of [5]:

**Corollary.** Any locally compact topological polycyclic monoid  $S$  is discrete. More precisely,  $S$  is topologically isomorphic to the topological  $\lambda$ -polycyclic monoid  $\mathcal{P}_\lambda^d$  for a unique cardinal  $\lambda \geq 2$ .

Some other topologizability results of the same flavor can be found in [24, 23, 17, 2, 20, 13, 4, 5, 6].

## PROOF OF MAIN THEOREM

The proof of Main Theorem is divided into a series of 12 lemmas.

Let  $S$  be a non-discrete locally compact semitopological polycyclic monoid and let  $\Lambda$  be its generating set. By [5, Proposition 2.2],  $S$  is algebraically isomorphic to the  $\lambda$ -polycyclic monoid  $\mathcal{P}_\lambda$  for a unique cardinal  $\lambda \geq 2$ . So, we can identify  $S$  with  $\mathcal{P}_\lambda$  and the cardinal  $\lambda$  with the generating set  $\Lambda$  of the inverse monoid  $S$ .

Let  $S^+$  be the submonoid of  $S$ , generated by the set  $\Lambda$  (i.e.,  $S^+$  is the smallest submonoid of  $S$  containing the generating set  $\Lambda$ ). Elements of  $S^+$  can be identified with words in the alphabet  $\Lambda$ . Such words will be called *positive*. The relations between the generators of  $S$  guarantee that each non-zero element  $a$  of  $S$  can be uniquely written as  $u^{-1}v$  for some positive words  $u, v \in S^+$ . Then by  $\downarrow a$  we denote the set of all prefixes of the word  $u^{-1}v$ . For a subset  $C \subset S$  we put  $\downarrow C = \bigcup_{a \in C} \downarrow a$ .

The following algebraic property of a polycyclic monoid is proved in [5, Proposition 2.7].

**Lemma 1.** *For any non-zero elements  $a, b, c \in S$ , the set  $\{x \in S : axb = c\}$  is finite.*

This lemma will be applied in the proof of the following useful fact, proved in [5, Proposition 3.1].

**Lemma 2.** *All non-zero elements of  $S$  are isolated points in the space  $S$ .*

*Proof.* For convenience of the reader we present a short proof of this important lemma. First we show that the unit 1 is an isolated point of the semitopological monoid  $S$ . Take any generator  $g \in \Lambda$  and consider the idempotent  $e = g^{-1}g$  of  $S$ . Since the map  $S \rightarrow eS$ ,  $x \mapsto ex$ , is a retraction of the Hausdorff space  $S$  onto  $eS$ , the principal right ideal  $eS = g^{-1}S$  is closed in  $S$ . By the same reason, the principal left ideal  $Se = Sg$  is closed in  $S$ . The separate continuity of the semigroup operation yields a neighborhood  $U_1 \subset S \setminus (g^{-1}S \cup Sg)$  of 1 such that  $0 \notin (e \cdot U_1) \cap (U_1 \cdot e)$ . We claim that  $U_1 = \{1\}$ . In the opposite case,  $U_1$  contains some element  $a \neq 1$ , which can be written as  $u^{-1}v$  for some positive words  $u, v \in S^+$ . Since  $a \neq 1$  one of the words  $u, v$  is not empty. If  $u$  is not empty, then  $a \in U_1 \subset S \setminus g^{-1}S$  implies that the word  $u^{-1}$  does not start with  $g^{-1}$ . In this case  $ea = g^{-1}gu^{-1}v = g^{-1} \cdot 0 = 0$ , which contradicts the choice of the neighborhood  $U_1 \ni a$ . If the word  $v$  is not empty, then  $a \in U_1 \subset S \setminus Sg$  implies that  $v$  does not end with  $g$ . In this case  $ae = u^{-1}vg^{-1}g = 0$ , again contradicting the choice of  $U_1$ . This contradiction shows that the unit 1 is an isolated point of  $S$ .

Now we can prove that each non-zero point  $a \in S$  is isolated. Write  $a$  as  $u^{-1}v$  for some positive words  $u, v \in S^+$ . Since  $uav^{-1} = 1$ , the separate continuity of the semigroup operation on  $S$ , yields an open neighborhood  $O_a \subset S$  of  $a$  such that  $uO_av^{-1} \subset U_1 = \{1\}$ . By Lemma 1, the neighborhood  $O_a$  is finite and hence the singleton  $\{a\} = O_a \setminus (O_a \setminus \{a\})$  is open, which means that the point  $a$  is isolated in  $S$ .  $\square$

Lemma 2 implies that the locally compact space  $S$  has a neighborhood base at zero, consisting of compact sets. It also implies the following useful lemma.

**Lemma 3.** *For any compact neighborhoods  $U_0, V_0 \subset S$  of zero the set  $U_0 \setminus V_0$  is finite.*

For an element  $u \in S$  by  $\mathcal{R}_u := \{x \in S : xS = uS\}$  we denote its *Green  $\mathcal{R}$ -class* in  $S$ . Here  $uS = \{us : s \in S\}$  is the right principal ideal generated by the element  $u$ .

**Lemma 4.** *Every non-zero  $\mathcal{R}$ -class in  $S$  coincides with the  $\mathcal{R}$ -class  $\mathcal{R}_{u^{-1}} = \mathcal{R}_{u^{-1}u}$  for some positive word  $u \in S^+$ .*

*Proof.* Each non-zero element of the semigroup  $\mathcal{P}_\lambda$  can be written as  $u^{-1}v$  for some positive words  $u, v \in S^+$ . Taking into account that  $u^{-1}v \cdot v^{-1} = u^{-1}$ , we conclude that  $\mathcal{R}_{u^{-1}v} = \mathcal{R}_{u^{-1}} = \mathcal{R}_{u^{-1}u}$ .  $\square$

In the following Lemmas 5–12 we assume that  $U_0$  is any fixed compact neighborhood of zero in the semitopological monoid  $S$ . Since zero is a unique non-isolated point in  $S$ , the neighborhood  $U_0$  is infinite.

**Lemma 5.** *The neighborhood  $U_0$  has infinite intersection with some  $\mathcal{R}$ -class of  $S$ .*

*Proof.* To derive a contradiction, assume  $U_0$  has finite intersection with each  $\mathcal{R}$ -class of the semigroup  $S$ . Taking into account that  $U_0$  is infinite and applying Lemma 4, we can see that the set  $B = \{u \in S^+ : \mathcal{R}_{u^{-1}} \cap U_0 \neq \emptyset\}$  is infinite. For every  $u \in B$  denote by  $v_u$  a longest positive word in  $S^+$  such that  $u^{-1}v_u \in \mathcal{R}_{u^{-1}} \cap U_0$  (such word  $v_u$  exists as the set  $\mathcal{R}_{u^{-1}} \cap U_0$  is finite). It follows that  $A = \{u^{-1}v_u : u \in B\}$  is an infinite subset of  $U_0$ . Fix any element  $g$  of the generating set  $\Lambda$  of  $S$ . Since  $0 \cdot g = 0$ , we can use the separate continuity of the semigroup operation of  $S$  and find a compact neighborhood  $V_0 \subseteq U_0$  of zero such that  $V_0 \cdot g \subseteq U_0$ . But then  $V_0 \subseteq U_0 \setminus A$  which contradicts Lemma 3.  $\square$

**Lemma 6.** *The neighborhood  $U_0$  has infinite intersection with each non-zero  $\mathcal{R}$ -class of the semigroup  $S$ .*

*Proof.* By Lemma 4, any non-zero  $\mathcal{R}$ -class of the semigroup  $S = \mathcal{P}_\lambda$  is of the form  $\mathcal{R}_{v^{-1}}$  for some positive word  $v \in S^+$ . By Lemmas 4 and 5, for some element  $u \in S^+$  the intersection  $U_0 \cap \mathcal{R}_{u^{-1}}$  is infinite. Observe that  $v^{-1}u \cdot \mathcal{R}_{u^{-1}} \subset \mathcal{R}_{v^{-1}}$ . By the separate continuity of the semigroup operation at  $0 = v^{-1}u \cdot 0$ , there exists a neighborhood  $V_0 \subset S$  of zero such that  $v^{-1}u \cdot V_0 \subset U_0$ . By Lemma 3, the difference  $U_0 \setminus V_0$  is finite, which implies that the intersection  $V_0 \cap \mathcal{R}_{u^{-1}}$  is infinite. Then the set  $v^{-1}u \cdot (V_0 \cap \mathcal{R}_{u^{-1}}) \subset U_0 \cap \mathcal{R}_{v^{-1}}$  is infinite, too.  $\square$

**Lemma 7.** *If the generating set  $\Lambda$  is finite, then the neighborhood  $U_0$  contains all but finitely many elements of the  $\mathcal{R}$ -class  $\mathcal{R}_1 = \{x \in S : xS = S\}$ .*

*Proof.* To derive a contradiction, assume that the set  $A := \mathcal{R}_1 \setminus U_0$  is infinite. We claim that for every  $g \in \Lambda$  the set  $A_g = \{a \in A : ag \in U_0\}$  is finite. Indeed, suppose that  $A_g$  is infinite. By Proposition 1,  $A_g \cdot g$  is an infinite subset of  $U_0$ . Since  $0 \cdot g^{-1} = 0$ , the separate continuity of the semigroup operation on  $S$  yields a compact neighborhood  $V_0 \subseteq U_0$  of zero such that  $V_0 \cdot g^{-1} \subseteq U_0$ . Then  $V_0 \subseteq U_0 \setminus (A_g \cdot g)$  which contradicts Lemma 3.

Let  $A^* = A \setminus \bigcup_{g \in \Lambda} \downarrow A_g$  (we recall that  $\downarrow A_g = \bigcup_{a \in A_g} \downarrow a$  where  $\downarrow a$  is the set of all prefixes of the word  $a$ ). It follows that  $A^*$  is a cofinite (and hence infinite) subset of  $A$ . Now we are going to show that  $A^*$  is a right ideal of  $\mathcal{R}_1$ . In the opposite case we could find elements  $c \in \mathcal{R}_1$  and  $v \in A^*$  such that  $vc \notin A^*$ . Let  $c^*$  be the longest prefix of  $c$  such that  $vc^* \in A^*$  (the word  $c^*$  can be empty, in which case it is the unit of  $S$ ). Then  $vc^*g \notin A^*$  for some  $g \in \Lambda$ . Observe that  $vc^* \in A^* \subset A \subset \mathcal{R}_1$  implies  $vc^*g \in \mathcal{R}_1$ .

Assuming that  $vc^*g \in U_0$ , we conclude that  $vc^* \in A_g \subset \downarrow A_g$ , which contradicts the inclusion  $vc^* \in A^*$ . So,  $vc^*g \notin U_0$  and hence  $vc^*g \in A$ . Then  $vc^*g \notin A^*$  implies that  $vc^*g \in \downarrow A_f$  for some  $f \in \Lambda$  and thus  $vc^* \in \downarrow A_f$ , too. But this contradicts the inclusion  $vc^* \in A^*$ . The obtained contradiction implies that  $A^*$  is a right ideal of  $\mathcal{R}_1$ .

Let  $u \in A^*$  be an arbitrary element. Since  $u \cdot 0 = 0$ , the separate continuity of the semigroup operation yields a compact neighborhood  $V_0 \subset U_0$  of zero such that  $u \cdot V_0 \subseteq U_0$ . Proposition 1 and Lemma 6 imply that  $u \cdot (V_0 \cap \mathcal{R}_1)$  is an infinite subset of  $A^* \cap U_0 \subset A \cap U_0$ . In particular,  $A \cap U_0$  is not empty, which contradicts the definition of the set  $A := \mathcal{R}_1 \setminus U_0$ .  $\square$

**Lemma 8.** *If the cardinal  $\lambda = |\Lambda|$  is finite, then the neighborhood  $U_0$  contains all but finitely many elements of any  $\mathcal{R}$ -class  $\mathcal{R}_x$ ,  $x \in S$ .*

*Proof.* The lemma is trivial if  $x = 0$ . So we assume that  $x \neq 0$ . By Lemma 4,  $\mathcal{R}_x = \mathcal{R}_{u^{-1}}$  for some positive word  $u \in S^+$ . Since  $u^{-1} \cdot 0 = 0$ , the separate continuity of the semigroup operation yields a neighborhood  $V_0 \subseteq U_0$  of zero such that  $u^{-1} \cdot V_0 \subseteq U_0$ . By Lemmas 3 and 7,  $\mathcal{R}_1 \subset^* V_0$  (which means that  $\mathcal{R}_1 \setminus V_0$  is finite). Then  $\mathcal{R}_x = \mathcal{R}_{u^{-1}} = u^{-1} \cdot \mathcal{R}_1 \subset^* u^{-1} \cdot V_0 \subset U_0$ , which means that  $U_0$  contains all but finitely many points of the  $\mathcal{R}$ -class  $\mathcal{R}_x$ .  $\square$

The following lemma proves Main Theorem in case of finite cardinal  $\lambda = |\Lambda|$ .

**Lemma 9.** *If the cardinal  $\lambda$  is finite, then the set  $S \setminus U_0$  is finite.*

*Proof.* To derive a contradiction, assume that  $S \setminus U_0$  is infinite. By Lemma 8, for each  $u \in S^+$  the set  $\mathcal{R}_{u^{-1}} \setminus U_0$  is finite. Since the complement  $S \setminus U_0 = \bigcup_{u \in S^+} \mathcal{R}_{u^{-1}} \setminus U_0$  is infinite, the set  $B = \{u \in S^+ : \mathcal{R}_{u^{-1}} \setminus U_0 \neq \emptyset\}$  is infinite, too. For every  $u \in B$  denote by  $v_u$  the longest word in  $S^+$  such that  $u^{-1}v_u \in \mathcal{R}_{u^{-1}} \setminus U_0$ . Then  $C = \{u^{-1}v_u : u \in B\} \subset \mathcal{R}_{u^{-1}} \setminus U_0$  is infinite and by Proposition 1, for every  $g \in \Lambda$  the set  $C \cdot g$  is an infinite subset of  $U_0$ . Since  $0 \cdot g^{-1} = 0$ , the separate continuity of the semigroup operation yields a neighborhood  $V_0 \subset U_0$  of zero such that  $V_0 \cdot g^{-1} \subseteq U_0$ . By Lemma 3, the set  $U_0 \setminus V_0$  is finite. Since the set  $Cg \subset U_0$  is infinite, there is an element  $c \in C$  with  $cg \in V_0$ . Then  $c = cgg^{-1} \in V_0g^{-1} \subset U_0$ , which contradicts the inclusion  $C \subset \mathcal{R}_1 \setminus U_0$ .  $\square$

**Lemma 10.** *The set  $\mathcal{R}_1 \setminus U_0$  is finite.*

*Proof.* To derive a contradiction, assume that the complement  $A := \mathcal{R}_1 \setminus U_0$  is infinite. By Lemma 6, the set  $U_0 \cap \mathcal{R}_1$  is infinite.

For a finite subset  $F \subset \Lambda$ , let  $S_F$  be the smallest subsemigroup of  $S$  containing the set  $F \cup F^{-1} \cup \{0, 1\}$ . If  $|F| \geq 2$ , then  $S_F$  is a polycyclic monoid. Separately, we shall consider two cases.

1. First assume that for every finite subset  $F \subset \Lambda$  the set  $U_0 \cap S_F$  is finite. In this case for every point  $g \in \Lambda$ , consider the set  $W_g = \{a \in U_0 \cap \mathcal{R}_1 : ag \notin U_0\}$ . The separate continuity of the semigroup operation yields a neighborhood  $V_0 \subset U_0$  of zero such that  $V_0 \cdot g \subset U_0$ . Lemma 3 implies that the set  $W_g \subset U_0 \setminus V_0$  is finite and hence for every non-empty finite subset  $F \subset \Lambda$  the set  $U_F := (U_0 \cap \mathcal{R}_1) \setminus \bigcup_{g \in F} W_g$  is infinite. We claim that  $U_F \cdot y \subseteq U_F$  for every  $y \in S_F \cap \mathcal{R}_1$ . In the opposite case, there exist elements  $y \in S_F \cap \mathcal{R}_1$  and  $x \in U_F$  such that  $xy \notin U_F$ . Let  $y^*$  be the longest prefix of  $y$  such that  $xy^* \in U_F$  (note that  $y^*$  could be equal to 1). Then  $xy^*g \notin U_F$  for some  $g \in F$ . Hence  $xy^* \in W_g$  which contradicts the definition of  $U_F \ni xy^*$ . Hence  $U_F \cdot y \subseteq U_F$  for each element  $y \in S_F \cap \mathcal{R}_1$ .

Fix any element  $v \in U_F$  and find a finite subset  $D \subset \Lambda$  such that  $v \in S_D$ ,  $F \subset D$  and  $|D| \geq 2$ . Proposition 1 implies that  $v \cdot (S_F \cap \mathcal{R}_1)$  is an infinite subset of  $U_F \cap S_D$ , which contradicts our assumption.

2. Next, assume that for some finite subset  $F \subset \Lambda$  the intersection  $U_0 \cap S_F$  is infinite. For every  $g \in F$  consider the subset  $A_g := \{a \in A : ag \in U_0\}$  of the infinite set  $A = \mathcal{R}_1 \setminus U_0$ . The separate continuity of the semigroup operation yields a neighborhood  $V_0 \subset S$  of zero such that  $V_0 \cdot g^{-1} \subset U_0$ . We claim that for every  $a \in A_g$  we get  $ag \notin V_0$ . In the opposite case we would get  $a = agg^{-1} \in V_0 \cdot g^{-1} \subset U_0$ , which contradicts the inclusion  $a \in A$ . Then  $A_g = \{a \in A : ag \in U_0 \setminus V_0\}$  and this set is finite by Lemmas 3 and 1. It follows that  $A_F = A \setminus \bigcup_{g \in F} \downarrow A_g$  is a cofinite (and hence infinite) subset of  $A$ .



We claim that  $A_F \cdot y \subseteq A_F$  for every  $y \in S_F \cap \mathcal{R}_1$ . In the opposite case, we can find elements  $y \in S_F \cap \mathcal{R}_1$  and  $x \in A_F$  such that  $xy \notin A_F$ . Let  $y^*$  be the longest prefix of  $y$  such that  $xy^* \in A_F$  (note that  $y^*$  could be equal to 1). Then  $xy^*g \notin A_F$  for some  $g \in F$ . It follows from  $xy^* \in A_F \subset A = \mathcal{R}_1 \setminus U_0$  and  $gg^{-1} = 1$  that  $xy^*g \in \mathcal{R}_1$ . Assuming that  $xy^*g \in U_0$ , we conclude that  $xy^* \in A_g$ , which contradicts the inclusion  $xy^* \in A_F$ . So,  $xy^*g \in \mathcal{R}_1 \setminus U_0 = A$  and then  $xy^*g \notin A_F$  implies that  $xy^*g \in \downarrow A_h$  for some  $h \in F$  and finally  $xy^* \in \downarrow A_h$ , which contradicts the inclusion  $xy^* \in A_F$ . This contradiction completes the proof of the inclusion  $A_F \cdot y \subseteq A_F$  for each  $y \in S_F \cap \mathcal{R}_1$ .

Fix any element  $v \in A_F$  and find a finite subset  $D \subset \Lambda$  such that  $v \in S_D$ ,  $F \subset D$  and  $|D| \geq 2$ . The subset  $S_D$  contains the unique non-isolated point of the space  $S$  and hence is closed in  $S$ . The local compactness of  $S$  implies the local compactness of the polycyclic monoid  $S_D$  endowed with the subspace topology. Lemma 3 and our assumption guarantee that the semitopological polycyclic monoid  $S_D$  is not discrete. By Proposition 1,  $v \cdot (S_F \cap \mathcal{R}_1)$  is an infinite subset of  $A_F \cap S_D \subset S_D \setminus U_0$ . But this contradicts Lemma 9 (applied to the locally compact polycyclic monoid  $S_D$  and the neighborhood  $U_0 \cap S_D$  of zero in  $S_D$ ).  $\square$

**Lemma 11.** *The neighborhood  $U_0$  contains all but finitely many points of each  $\mathcal{R}$ -class in  $S$ .*

*Proof.* By Lemma 4, it suffices to check that for any  $u \in S^+$  the set  $\mathcal{R}_{u^{-1}} \setminus U_0$  is finite. The separate continuity of the semigroup operation yields a compact neighborhood  $V_0 \subseteq U_0$  of zero such that  $u^{-1} \cdot V_0 \subseteq U_0$ . By Lemmas 10 and 3, we get  $\mathcal{R}_1 \subset^* V_0$ . Then  $\mathcal{R}_{u^{-1}} = u^{-1} \cdot \mathcal{R}_1 \subset^* u^{-1} \cdot V_0 \subset U_0$ , which means that the set  $\mathcal{R}_{u^{-1}} \setminus U_0$  is finite.  $\square$

Our final lemma combined with Lemma 2 proves Main Theorem and shows that the semitopological polycyclic monoid  $S$  carries the topology of one-point compactification of the discrete space  $S \setminus \{0\}$ .

**Lemma 12.** *The complement  $S \setminus U_0$  is finite and hence  $S$  is compact.*

*Proof.* To derive a contradiction, assume that the set  $S \setminus U_0$  is infinite. By Lemma 11, for each  $u \in S^+$  the set  $\mathcal{R}_{u^{-1}} \setminus U_0$  is finite. Since  $S = \bigcup_{u \in S^+} \mathcal{R}_{u^{-1}}$ , the set  $B = \{u \in S^+ : \mathcal{R}_{u^{-1}} \setminus U_0 \neq \emptyset\}$  is infinite. For every  $u \in B$  denote by  $v_u$  the longest word in  $S^+$  such that  $u^{-1}v_u \in \mathcal{R}_{u^{-1}} \setminus U_0$ . Then  $C = \{u^{-1}v_u : u \in B\}$  is an infinite subset of  $S \setminus U_0$ . By Lemma 1, for any  $g \in \Lambda$  the set  $C \cdot g$  is infinite. The separate continuity of the semigroup operation yields a neighborhood  $V_0 \subset U_0$  of zero such that  $V_0 \cdot g^{-1} \subset U_0$ . Then  $V_0 \subset U_0 \setminus (C \cdot g)$  which contradicts Lemma 3.  $\square$

## ACKNOWLEDGEMENTS

The author acknowledges professors Taras Banakh and Oleg Gutik for their fruitful comments and suggestions.

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